10/22 Recitation

1. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a set of nonzero orthogonal vectors. Show that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent.

   Hint: Take the dot product of each $\mathbf{v}_i$ with the equation $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x \end{bmatrix}$.

   (a) Find the matrix associated to T and show that it is orthogonal.

   (b) Evaluate T at the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and sketch the original vectors and images in $\mathbb{R}^2$.

   (c) What does T do to these vectors, geometrically? Use this intuition to explain why T preserves lengths and angles.

3. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{bmatrix}$.

   (a) Use Gram-Schmidt to find an orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2\}$ of $C(A)$.

   (b) Use part (a) to find the QR factorization of A.

   (c) Use the QR factorization to find a least squares solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

4. **Programming Challenge**: Write a function in Julia that implements Gram-Schmidt. The input should be a matrix A whose columns form a basis for a vector space V and the output should be a matrix Q whose columns form an orthonormal basis for V.